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LETTER TO THE EDITOR

Fermion mixing in quasifree states**K C Hannabuss and D C Latimer¹**

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Online at stacks.iop.org/JPhysA/36/L69**Abstract**

Quantum field-theoretic treatments of fermion oscillations are typically restricted to calculations in Fock space. In this letter, we extend the oscillation formulae to include more general quasifree states, and also consider the case when the mixing is not unitary.

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1. Introduction

The theoretical underpinning for fermion oscillations was developed decades ago [8], but only recently have terrestrial and solar neutrino experiments begun to substantiate this work [32, 33], and experimental evidence for oscillations has led to renewed interest in the theory. Current quantum field-theoretic treatments of the fermion oscillation phenomena [1, 2, 9–19, 23, 31] have brought about modifications to the oscillation formula developed in [8], adding terms depending on the sum of energies rather than their difference. Although these new results are based on calculations made in fermion Fock space, they rely on modifications suggested by physical considerations. In this letter, we generalize the oscillation formula to general quasifree states, and show that the additional terms occur naturally in that setting. Our result contains the known formula for Fock states as a special case, but also includes other physical scenarios such as the thermal (KMS) state, or situations in which polarization of the vacuum has occurred. The oscillation formula for the thermal state could be a better approximation for fermions at nonzero temperature in some extreme physical situations, such as the early universe. At the end of the letter, we shall also consider the case when the mixing operator is not unitary and show that it leads to similar effects.

From a mathematical point of view the main obstacle to such calculations lies in the fact that we wish to calculate the oscillatory behaviour of correlations between projections onto flavour states (such as the ν_e and ν_μ states) at different times, but the dynamical behaviour is simplest in states with definite masses. These are distinct from the states of definite flavour to which they are related by a non-trivial *mixing* transformation.

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The fermionic anticommutation relations (CAR) can be written in terms of smeared creation operators

$$c(w) = \int w(x)a^*(x) dx$$

(smeared with test functions w, z in the complex inner product space of wavefunctions on \mathbf{R}^3 with values in the product of Dirac spinors \mathcal{V} and an N -dimensional space V describing the various flavour states) as

$$[c(w)^*, c(z)]_+ = \langle w, z \rangle \quad [c(w), c(z)]_+ = 0.$$

It is well known [25] that the (abstract) algebra defined by these relations has many inequivalent representations by operators in Hilbert space, and the study of the interrelations between a selection of these forms the main focus of this letter. We shall mainly be concerned with quasifree representations of the CAR algebra which generalize the standard Fock and Dirac–Fock representations. (The Fock representations themselves have been studied, for example, in [9, 10, 29, 34].) Quasifree representations are those in which the Wick determinant formula expresses the n -point correlation functions in terms of the two-point correlation functions just as in Fock space. As well as appearing for thermal states of systems, quasifree states often arise in situations where the vacuum is polarized, and so allow us to treat more complicated field-theoretic effects whilst avoiding the detailed models. A well-known technique of Powers and Størmer [35] and Araki [3] tells us how to construct any quasifree representation of $\text{CAR}(W)$ as the composition of an injection of W into $W \oplus W$ with a Fock representation of $\text{CAR}(W \oplus W)$, and so we shall concentrate on Fock representations.

This letter presents a purely algebraic field-theoretic treatment of neutrino mixing. There are alternative approaches motivated by an argument that inequivalent representations produce differences too small to yield observable effects in current experiments, in particular, those which include model sources and detectors, often using wave packets [6, 7, 20–22, 24, 26–28, 30]. We believe that it should also be possible to include models of the source and detector within our field-theoretic treatment and hope to return to that at a future date. (We also note that Fell’s theorem [24] says that the correlation functions in any representation can be approximated arbitrarily closely by an appropriate state in a fixed faithful representation. Unlike Fock states, most of the quasifree states which we shall discuss are faithful.)

2. The one-particle space

To establish notation we first recall that for Dirac particles the elements of $W = W_m$ can be thought of as the initial data for the Dirac equation

$$i\hbar \frac{\partial w}{\partial t} = H_D w$$

where the Dirac Hamiltonian H_D is given in terms of momentum operators \mathbf{P} by

$$H_D = c(\alpha \otimes \mathbf{P} + \beta \otimes Mc)$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β satisfying the Clifford algebra relations

$$\alpha_j \beta + \beta \alpha_j = 0 \quad \beta^2 = 1 \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \quad j, k = 1, 2, 3$$

and M a positive operator on V whose eigenvalues give the masses m_1, m_2, \dots, m_n . (In what follows, we shall omit the tensor product and write βM for $\beta \otimes M$, etc.)

Choosing a basis in which the mass matrix M is diagonal, we refer to the solutions of the Dirac equation as mass eigenstates. We define

$$P_{\pm} = \frac{1}{2}(1 \pm H_D E^{-1})$$

where E is the positive square root of the positive operator $H_D^2 = (|\mathbf{P}|^2 + M^2 c^4)$. (We shall also use E_j for the value when in the eigenstate with mass m_j .) The P_{\pm} are idempotent, self-adjoint and $P_{\pm} H_D = \pm E P_{\pm}$; that is, they are the positive and negative energy projections on W . They also determine the *mass representation* of the CAR algebra with creation operators c_m and a Dirac–Fock vacuum vector Ω_m which satisfies the Dirac condition that, for every w in W ,

$$c_m(P_+ w)^* \Omega_m = 0 = c_m(P_- w) \Omega_m.$$

When we study the flavour space W_f we work rather with a *flavour representation* c_f in a standard Fock space with a flavour vacuum Ω_f satisfying $c_f(w)^* \Omega_f = 0$ for all $w \in W_f$.

3. General mixing transformations

In our previous letter, we studied what happened when the mass and flavour spaces W_m and W_f were isomorphic by a unitary mixing transformation T , but here we shall consider more general situations such as orthogonal mixing transformations, and Powers–Størmer transformations which enable us to realize quasifree states on Fock spaces. When T is unitary the spaces W_m and W_f can be identified, but, for orthogonal T , when W_m and W_f are the same as real spaces but have different complex structures, it is simpler to treat them as distinct.

It will be convenient to consider more generally the case when we have two inner product spaces W_j and W_k and a map $T_{jk} : W_k \rightarrow W_j$ which is orthogonal in the sense that it preserves the real part of the inner product: for all $w, z \in W_k$,

$$\langle z, w \rangle + \langle w, z \rangle = \langle T_{jk} z, T_{jk} w \rangle + \langle T_{jk} w, T_{jk} z \rangle.$$

For any real-linear operator T_{jk} on W_k we define the complex linear map $a_{jk} = \frac{1}{2}(T_{jk} - J_j T_{jk} J_k)$ and the antilinear map $b_{jk} = \frac{1}{2}(T_{jk} + J_j T_{jk} J_k)$ where J_j and J_k simply indicate multiplication by i on W_j and W_k , respectively. The Fock space creation and annihilation operators c_j and c_k are linked by

$$c_k(w) = c_j(a_{jk} w) + c_j(b_{jk} w)^*$$

and

$$c_j(T_{jk} w) + c_j(T_{jk} w)^* = c_j(a_{jk} w) + c_j(b_{jk} w)^* + c_j(a_{jk} w)^* + c_j(b_{jk} w)$$

for all $w \in W_k$, where a_{jk} and b_{jk} are the Bogoliubov maps just defined. To be consistent with the anticommutation relations in both W_j and W_k we have the orthogonality relations

$$a_{jk}^* a_{jk} + b_{jk}^* b_{jk} = 1 = a_{jk} a_{jk}^* + b_{jk} b_{jk}^* \quad a_{jk}^* b_{jk} + b_{jk}^* a_{jk} = 0 = a_{jk} b_{jk}^* + b_{jk} a_{jk}^*.$$

These are just the conditions that T be orthogonal. Since these transformations mix creation and annihilation operators it is expedient to introduce a more succinct notation. We combine creators and annihilators in the row vector $\tilde{c}_j = (c_j \ c_j^*)$, and introduce

$$\Lambda_{jk} = \begin{pmatrix} a_{jk} & b_{jk} \\ b_{jk} & a_{jk} \end{pmatrix}$$

to obtain

$$\tilde{c}_k(\tilde{w}) = \tilde{c}_j(\Lambda_{\cdot jk} \tilde{w}) \quad \text{for } \tilde{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

One advantage of dealing with the general situation is that there is an obvious composition law

$$\Lambda_{jl} = \Lambda_{jk} \Lambda_{kl}$$

obtained by writing the relationship between creation operators on W_j and W_l directly and through the intermediate space W_k . The orthogonality properties of the Bogoliubov maps can also be interpreted as telling us that $a_{jk} = a_{kj}^*$, and $b_{jk} = b_{kj}^*$, or

$$\Lambda_{jk} = \Lambda_{kj}^*.$$

(This is closely related to Araki's self-dual construction [3].)

Instead of the vacuum states described above we shall use a more general quasifree state ω for the CAR algebra. As noted above, this is determined by the two-point correlation functions which define a complex linear operator R and a conjugate linear operator S by

$$\omega[c(w)^*c(z)] = \langle w, Rz \rangle \quad \omega[c(w)c(z)] = \langle Sw, z \rangle$$

where $R = R^*$, $0 \leq R \leq 1$ and $S = -S^*$, [3–5]. We note that if $K = i(2R - 2S - 1)$ defines a complex structure on W , that is $K^2 = -1$, then the state ω is a Fock state for some choice of complex structure.

The GNS construction guarantees the existence of a representation $\pi : \text{CAR}(W) \rightarrow \mathcal{H}$ containing a cyclic vector $\Omega \in \mathcal{H}$ such that $\omega(b) = \langle \Omega, \pi(b)\Omega \rangle$ for any $b \in \text{CAR}(W)$. The representation π over \mathcal{H} can be expressed in terms of a Dirac–Fock representation of $\text{CAR}(W_+ \oplus W_-)$, where W_+ and W_- are both isomorphic to W . The representation in question takes $(w_+, w_-) \in W_+ \oplus W_-$ to $c_+(w_+) + c_-(w_-)$ and the w_+ with a Dirac-type vacuum Ω which is killed by the annihilators $c_+(w_+)^*$, and the creators $c_-(w_-)$. This means that

$$\langle c_+(w)\Omega, c_+(z)\Omega \rangle = \langle w, z \rangle \quad \langle c_-(w)^*\Omega, c_-(z)^*\Omega \rangle = \langle z, w \rangle$$

and all other two-point correlation functions vanish. Suppressing the representation map, and writing $c = \pi \circ c$, the required representation of $\text{CAR}(W)$ is given by

$$c(w) = \frac{1}{\sqrt{2}}(c_+(a_+w) + c_+(b_+w)^* + c_-(a_-w) + c_-(b_-w)^*)$$

where a_{\pm} are linear and b_{\pm} are antilinear and satisfy the orthogonality relations $a_{\pm}^*a_{\pm} + b_{\pm}^*b_{\pm} = 1$, $a_{\pm}^*b_{\pm} + b_{\pm}^*a_{\pm} = 0$. (It is easy to check that this does provide a representation of $\text{CAR}(W)$.)

When $W = W_j$ we shall write $a_{\pm j}$ and $b_{\pm j}$ for the Bogoliubov maps to give

$$c_j(w) = \frac{1}{\sqrt{2}}(c_+(a_{+j}w) + c_+(b_{+j}w)^* + c_-(a_{-j}w) + c_-(b_{-j}w)^*).$$

We adopt the summation convention that when an index s can range over different values $\{+, -\}$ and is repeated (as in expressions such as $c_s(a_{sj}w)$ or $c_j(a_{js}a_{sk}w)$) one sums over all its values and divides by $\sqrt{2}$. Then the above expression can be abbreviated to $\tilde{c}_j(\tilde{w}) = \tilde{c}_s(\Lambda_{sj}\tilde{w})$, and it is easy to see that our earlier rules for compositions apply. Thus a state which gives a quasifree representation for the label j does the same for the label k .

With this notation we have $\tilde{c}_j(w) = \tilde{c}_s(\Lambda_{sj}w)$. When applied to Ω , only certain of the components are nonzero. For instance,

$$\begin{aligned} c_j(w)\Omega &= \frac{1}{\sqrt{2}}(c_+(a_{+j}w) + c_+(b_{+j}w)^* + c_-(a_{-j}w) + c_-(b_{-j}w)^*)\Omega \\ &= \frac{1}{\sqrt{2}}(c_+(a_{+j}w) + c_-(b_{-j}w)^*)\Omega \end{aligned}$$

and, similarly,

$$c_j(w)^*\Omega = \frac{1}{\sqrt{2}}(c_+(b_{+j}w) + c_-(a_{-j}w)^*)\Omega.$$

Reversing the expansion we obtain

$$c_j(w)\Omega = \frac{1}{2}(c_j((a_{+j}a_{+j} + b_{-j}b_{-j})w) + c_j((a_{-j}b_{-j} + b_{+j}a_{+j})w)^*)\Omega$$

and

$$c_j(w)^*\Omega = \frac{1}{2}(c_j((a_{+j}b_{+j} + b_{-j}a_{-j})w) + c_j((a_{-j}a_{-j} + b_{+j}b_{+j})w)^*)\Omega$$

which may be combined as

$$\tilde{c}_j(\tilde{w})\Omega = \tilde{c}_j(\Lambda_j^0 \tilde{w})\Omega \quad \text{where} \quad \Lambda_j^0 = \begin{pmatrix} R & S^* \\ S & R' \end{pmatrix}$$

where $R = \frac{1}{2}(a_{j+}a_{+j} + b_{j-}b_{-j})$, $S = \frac{1}{2}(a_{j+}b_{+j} + b_{j-}a_{-j})$, $S^* = \frac{1}{2}(a_{j-}b_{-j} + b_{j+}a_{+j})$ (which the orthogonality relations show is the adjoint of S) and $R' = \frac{1}{2}(a_{j-}a_{-j} + b_{j+}b_{+j}) = 1 - R$ (by the orthogonality relations). In effect Λ_j^0 is the projection on W_j associated with Ω . We easily calculate that

$$\langle c_j(z)\Omega, c_j(w)\Omega \rangle = \langle c_j(z)\Omega, (c_j(Rw) + c_j(Sw)^*)\Omega \rangle = \langle z, Rw \rangle$$

so that R is the correlation operator already introduced, and similarly for S .

Whenever U is a unitary transformation of W_j , there is a unitary map $\pi_j(U)$ implementing U in the sense that

$$\pi_j(U)c_j(w)\pi_j(U)^{-1} = c_j(Uw)$$

for all $w \in W_j$. Since $\pi_j(U)$ is unitary there is a similar relation for the annihilator $c_j(w)^*$ and we combine these as

$$\pi_j(U)\tilde{c}_j(\tilde{w})\pi_j(U)^{-1} = \tilde{c}_j(\tilde{U}\tilde{w})$$

where

$$\tilde{U} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}.$$

(When j refers to flavour this gives a representation of the group of flavour transformations.)

We shall also need the infinitesimal version of this which arises from taking the unitary $\exp(isP)$ and differentiating at $s = 0$ to obtain

$$[\pi'_j(P), c_j(w)] = -i \frac{d}{ds} \pi_j(e^{isP})c_j(w)\pi_j(e^{isP})^{-1} |_{s=0} = -i \frac{d}{ds} c_j(e^{isP}w) |_{s=0} = c_j(Pw).$$

When $P = 1$, we can regard $N = \pi'_f(1)$ as the number operator, and when $P = P^\lambda$, the projection onto the states of flavour λ , then $N_\lambda = \pi'_f(P^\lambda)$ counts the number of flavour λ particles.

For our calculations we need to know the effect of $\pi_j(U)$ on $c_k(w)$ for different j and k . We, therefore, note that

$$\begin{aligned} \pi_j(U)\tilde{c}_k(\tilde{w})\pi_j(U)^{-1} &= \pi_j(U)\tilde{c}_j(\Lambda_{jk}\tilde{w})\pi_j(U)^{-1} \\ &= \tilde{c}_j(\tilde{U}\Lambda_{jk}\tilde{w}) \\ &= \tilde{c}_k(\Lambda_{kj}\tilde{U}\Lambda_{jk}\tilde{w}) \\ &= c_k((a_{kj}Ua_{jk} + b_{kj}Ub_{jk})w) + c_k((b_{kj}Ua_{jk} + a_{kj}Ub_{jk})w)^*. \end{aligned}$$

It is convenient to introduce the abbreviation $\tilde{U}_k = \Lambda_{kj}\tilde{U}\Lambda_{jk}$, and write

$$\pi_j(U)\tilde{c}_k(\tilde{w})\pi_j(U)^{-1} = \tilde{c}_k(\tilde{U}_k\tilde{w})$$

(it being understood that U is an operator on W_k). We have the explicit formula

$$\tilde{U}_k = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix}$$

with

$$u_k = a_{kj}Ua_{jk} + b_{kj}Ub_{jk} \quad v_k = a_{kj}Ub_{jk} + b_{kj}Ua_{jk}.$$

4. Correlation functions

The correlation functions which we first wish to calculate have the form

$$\langle c_f(z)\Omega, \pi_m(U)^* \pi'_f(P) \pi_m(U) c_f(w)\Omega \rangle$$

where P projects onto a flavour state, $U = \exp(-itH_D/\hbar)$ gives the time evolution and Ω defines a quasifree state. This can be easily found by differentiating the more tractable

$$\langle c_f(z)\Omega, \pi_m(U)^* \pi_f(D) \pi_m(U) c_f(w)\Omega \rangle$$

where $D = \exp(isP)$.

Gathering together our various comments we calculate that

$$\begin{aligned} \pi_m(U)^* \pi_f(D) \pi_m(U) \tilde{c}_f(\tilde{w})\Omega &= \pi_m(U)^* \pi_f(D) \pi_m(U) \tilde{c}_f(\Lambda_f^0 \tilde{w})\Omega \\ &= \tilde{c}_f(\tilde{U}_f^* \tilde{D} \tilde{U}_f \Lambda_f^0 \tilde{w}) \pi_m(U)^* \pi_f(D) \pi_m(U) \Omega. \end{aligned}$$

We now differentiate this to get

$$\begin{aligned} i\pi_m(U)^* \pi'_f(P) \pi_m(U) c_f(w)\Omega &= \tilde{c}_f(\tilde{U}_f^* i\tilde{P} \tilde{U}_f \Lambda_f^0 \tilde{w})\Omega + i\tilde{c}_f(\tilde{U}_f^* \tilde{U}_f \Lambda_f^0 \tilde{w}) \pi_m(U)^* \pi'_f(P) \pi_m(U) \Omega \\ &= \tilde{c}_f(\Lambda_f^0 \tilde{U}_f^* i\tilde{P} \tilde{U}_f \Lambda_f^0 \tilde{w})\Omega + i\tilde{c}_f(\Lambda_f^0 \tilde{w}) \pi_m(U)^* \pi'_f(P) \pi_m(U) \Omega. \end{aligned}$$

Inserting this expression into the inner product, but with $\tilde{w} = \begin{pmatrix} w \\ 0 \end{pmatrix}$, we obtain

$$\begin{aligned} \langle c_f(z)\Omega, \pi_m(U)^* \pi'_f(P) \pi_m(U) \tilde{c}_f(\tilde{w})\Omega \rangle &= -i\langle c_f(z)\Omega, \tilde{c}_f(\Lambda_f^0 \tilde{U}_f^* i\tilde{P} \tilde{U}_f \Lambda_f^0 \tilde{w})\Omega \rangle \\ &\quad + \langle c_f(z)\Omega, \tilde{c}_f(\Lambda_f^0 \tilde{w}) \pi_m(U)^* \pi'_f(P) \pi_m(U) \Omega \rangle. \end{aligned}$$

In each inner product we take the creation operators from the right to an adjoint acting on the left. There the factor of Λ_f^0 ensures that the adjoint annihilates Ω , so that we simply get

$$\begin{aligned} \langle c_f(z)\Omega, \pi_m(U)^* \pi'_f(P) \pi_m(U) \tilde{c}_f(\tilde{w})\Omega \rangle &= -i\langle [\tilde{c}_f(\Lambda_f^0 \tilde{U}_f^* i\tilde{P} \tilde{U}_f \Lambda_f^0 \tilde{w})^*, c_f(z)]_+ \Omega, \Omega \rangle \\ &\quad + \langle [\tilde{c}_f(\Lambda_f^0 \tilde{w})^*, c_f(z)]_+ \Omega, \pi_m(U)^* \pi'_f(P) \pi_m(U) \Omega \rangle. \end{aligned}$$

The anticommutators can now be written explicitly in terms of inner products. For example, the second gives

$$[\tilde{c}_f(\Lambda_f^0 \tilde{w})^*, c_f(z)]_+ = [c_f(Rw)^* + c_f(Sw), c_f(z)]_+ = \langle Rw, z \rangle.$$

The first requires a more detailed calculation, but we require only the first entry in $\Lambda_f^0 \tilde{U}_f^* i\tilde{P} \tilde{U}_f \Lambda_f^0 \tilde{w}$:

$$\begin{aligned} (1 \ 0) \begin{pmatrix} R & S^* \\ S & R' \end{pmatrix} \begin{pmatrix} u_f^* & v_f^* \\ v_f^* & u_f^* \end{pmatrix} \begin{pmatrix} iP & 0 \\ 0 & iP \end{pmatrix} \begin{pmatrix} u_f & v_f \\ v_f & u_f \end{pmatrix} \begin{pmatrix} R & S^* \\ S & R' \end{pmatrix} \begin{pmatrix} w \\ 0 \end{pmatrix} \\ = (R \ S^*) \begin{pmatrix} u_f^* & v_f^* \\ v_f^* & u_f^* \end{pmatrix} \begin{pmatrix} iP & 0 \\ 0 & iP \end{pmatrix} \begin{pmatrix} u_f & v_f \\ v_f & u_f \end{pmatrix} \begin{pmatrix} Rw \\ Sw \end{pmatrix}. \end{aligned}$$

Recalling that v is conjugate linear, the product of the three middle matrices can be written as

$$\begin{pmatrix} u_f^* & v_f^* \\ v_f^* & u_f^* \end{pmatrix} \begin{pmatrix} iP & 0 \\ 0 & iP \end{pmatrix} \begin{pmatrix} u_f & v_f \\ v_f & u_f \end{pmatrix} = i \begin{pmatrix} F & G \\ G & F \end{pmatrix}$$

where

$$F = u_f^* P u_f - v_f^* P v_f \quad G = u_f^* P v_f - v_f^* P u_f.$$

Using the fact that $R = R^*$ and recalling the conjugate linearity of S , this enables us to rewrite the first commutator as

$$(R \quad S^*) i \begin{pmatrix} F & G \\ G & F \end{pmatrix} \begin{pmatrix} R w \\ S w \end{pmatrix} = i(R^* F R + R^* G S - S^* G R - S^* F S) w.$$

Combining the expressions for the two commutators we obtain

$$\langle z, (R^* F R + R^* G S - S^* G R - S^* F S) w \rangle + \langle z, R w \rangle \langle \Omega, \pi_m(U)^* \pi'_f(P) \pi_m(U) \Omega \rangle.$$

The physically interesting quantity is the expected number of flavour λ particles $\langle N_\lambda(t) \rangle_\mu$, in a state where one flavour μ particle has been created a time t earlier out of the ‘vacuum’ Ω . This can be obtained by taking $P = P^\lambda$, $z = w = P^\mu \phi_j$, $U = V_t = \exp(-iH_D t/\hbar)$ above, and then summing as ϕ_j runs over an orthonormal basis of W , to get (with F^λ and G^λ denoting F and G when we take $P = P^\lambda$)

$$\sum_j \langle c_f(P^\mu \phi_j) \Omega, \pi_m(V_t)^* \pi'_f(P) \pi_m(V_t) c_f(P^\mu \phi_j) \Omega \rangle = \text{tr}[(R^* F^\lambda R + R^* G^\lambda S - S^* G^\lambda R - S^* F^\lambda S) P^\mu] + \text{tr}[R P^\mu] \langle \Omega, \pi_m(V_t)^* \pi'_f(P^\lambda) \pi_m(V_t) \Omega \rangle.$$

The last inner product is just the vacuum expectation of P^λ and should be subtracted (since we are only interested in the enhancement produced by creating a flavour state), and we must also divide by the norm of the state $\text{tr}[R P^\mu]$ to get

$$\langle N_\lambda(t) \rangle_\mu = \frac{\text{tr}[(R^* F^\lambda R + R^* G^\lambda S - S^* G^\lambda R - S^* F^\lambda S) P^\mu]}{\text{tr}[R P^\mu]}.$$

The expected total flavour number $\langle N(t) \rangle_\mu$ is obtained by summing over λ (which means that F^λ and G^λ are replaced by $u_f^* u_f - v_f^* v_f$ and $u_f^* v_f - v_f^* u_f$, respectively). The ratio $\langle N_\lambda(t) \rangle_\mu / \langle N(t) \rangle_\mu$ then gives the proportion of flavour λ particles. In the next two sections we shall look at two special cases of this formula.

During the preparation of this letter an interesting preprint appeared [19] which investigates the CP violation in three flavour mixing. We note that in our context the transition probability for antiparticles can be calculated by using annihilation in place of creation operators, which leads to the replacement of R by $R' = 1 - R$ and of S by $S^* = -S$, to give

$$\langle N_\lambda(t) \rangle_\mu = \frac{\text{tr}[\{(1 - R)^* F^\lambda (1 - R) - (1 - R)^* G^\lambda S + S^* G^\lambda (1 - R) - S^* F^\lambda S\} P^\mu]}{\text{tr}[R' P^\mu]}.$$

5. Unitary mixing in quasifree states

The first case which we shall consider is for a unitary mixing matrix but in a quasifree state. When the mixing transformation is given by a unitary operator T , we have $u_f = T U T^*$ and $v_f = 0$, so that $G^\lambda = 0$ and $F^\lambda = T V_t^* T^* P^\lambda T V_t T^*$. The expression then simplifies to

$$\langle N_\lambda(t) \rangle_\mu = \frac{\text{tr}[(R^* T V_t^* T^* P^\lambda T V_t T^* R - S^* T V_t^* T^* P^\lambda T V_t T^* S) P^\mu]}{\text{tr}[R P^\mu]}.$$

(This formula can be easily checked in the case of a thermal state ω_β at absolute temperature $(k\beta)^{-1}$, where k is Boltzmann’s constant (see the appendix), and gives the known values $R = (1 + e^{-\beta H_D})^{-1}$ and $S = 0$, in agreement with our general formula.)

We note that summation over λ gives

$$\langle N(t) \rangle_\mu = \frac{\text{tr}[(R^* T V_t^* T^* T V_t T^* R - S^* T V_t^* T^* T V_t T^* S) P^\mu]}{\text{tr}[R P^\mu]} = \frac{\text{tr}[(R^* R - S^* S) P^\mu]}{\text{tr}[R P^\mu]}$$

which is, as one would hope, independent of time.

In fact for states invariant under global $U(1)$ transformations of W , we always have $S = 0$ [4, 5], and then

$$\langle N_\lambda(t) \rangle_\mu = \frac{\text{tr}[R^* T V_t^* T^* P^\lambda T V_t T^* R P^\mu]}{\text{tr}[R P^\mu]}.$$

To achieve a more explicit formula we take $R = T\rho(H_D)T^*$ to be a function of the Hamiltonian, where $\rho(x)$ is a real function of the real variable x , defined everywhere except, perhaps, zero. Both the Dirac and KMS states satisfy this restriction. This gives

$$\langle N_\lambda(t) \rangle_\mu = \frac{\text{tr}[T\rho(H_D)V_t^*T^*P^\lambda TV_t\rho(H_D)T^*P^\mu]}{\text{tr}[T\rho T^*P^\mu]} = \frac{\text{tr}[\rho(H_D)V_t^*T^*P^\lambda TV_t\rho(H_D)T^*P^\mu T]}{\text{tr}[\rho T^*P^\mu T]}.$$

Performing some preliminary calculations, we have

$$\begin{aligned} \rho(H_D)V_t^* &= \rho(H_D)e^{iH_D t/\hbar}(P_+ + P_-) \\ &= \rho(E)e^{iEt/\hbar}P_+ + \rho(-E)e^{-iEt/\hbar}P_- \\ &= \frac{1}{2}[\rho(E)e^{iEt/\hbar} + \rho(-E)e^{-iEt/\hbar}] + \frac{1}{2}[\rho(E)e^{iEt/\hbar} - \rho(-E)e^{-iEt/\hbar}](P_+ - P_-) \end{aligned}$$

and, in particular,

$$T^*RT = \frac{1}{2}[\rho(E) + \rho(-E)] + \frac{1}{2}[\rho(E) - \rho(-E)](P_+ - P_-).$$

To condense our notation, we define

$$\begin{aligned} \sigma_j &= \rho(E_j)e^{iE_j t/\hbar} + \rho(-E_j)e^{-iE_j t/\hbar} \\ \delta_j &= \rho(E_j)e^{iE_j t/\hbar} - \rho(-E_j)e^{-iE_j t/\hbar} \\ \gamma_j &= \rho(E_j) + \rho(-E_j) \\ \epsilon_j &= c(\alpha \cdot \mathbf{P} + \beta m_j c)/E_j. \end{aligned}$$

We note that ϵ_j has trace zero but

$$\text{tr}(\epsilon_j \epsilon_k) = (|\mathbf{P}|^2 c^2 + m_j m_k c^4)/(E_j E_k) = S_{jk}.$$

We also know that in terms of the mass basis the components of P^λ are

$$(T^*P^\lambda T)_{jk} = T_{j\lambda}^* T_{\lambda k} = \bar{T}_{\lambda j} T_{\lambda k}.$$

With this notation the numerator in the oscillation formula is

$$\begin{aligned} \frac{1}{4} \sum_{j,k=1}^N \text{tr}[(\sigma_j + \delta_j \epsilon_j) T^* P^\lambda T (\bar{\sigma}_k + \bar{\delta}_k \epsilon_k)^* T^* P^\mu T] &= \sum_{j,k=1}^N [\sigma_j \bar{\sigma}_k + \delta_j \bar{\delta}_k S_{jk}] \bar{T}_{\lambda j} T_{\lambda k} \bar{T}_{\mu k} T_{\mu j} \\ &= \frac{1}{2} \sum_{j,k=1}^N [(\sigma_j \bar{\sigma}_k + \delta_j \bar{\delta}_k)(1 + S_{jk}) + (\sigma_j \bar{\sigma}_k - \delta_j \bar{\delta}_k)(1 - S_{jk})] \bar{T}_{\lambda j} T_{\lambda k} \bar{T}_{\mu k} T_{\mu j}. \end{aligned}$$

Now, recalling that ρ is a real function, we have

$$\begin{aligned} \sigma_j \bar{\sigma}_k + \delta_j \bar{\delta}_k &= 2[\rho(E_j)\rho(E_k)e^{i(E_j-E_k)t/\hbar} + \rho(-E_j)\rho(-E_k)e^{i(E_k-E_j)t/\hbar}] \\ \sigma_j \bar{\sigma}_k - \delta_j \bar{\delta}_k &= 2[\rho(E_j)\rho(-E_k)e^{i(E_j+E_k)t/\hbar} + \rho(-E_j)\rho(E_k)e^{-i(E_j+E_k)t/\hbar}]. \end{aligned}$$

When $m_j = m_k$ we have $S_{jk} = 1$ so that the second term in the numerator disappears, and since also $E_j = E_k$, we see that $\sigma_j \bar{\sigma}_k + \delta_j \bar{\delta}_k$ is time independent, and consequently there is no flavour oscillation between these. For Fock states and general masses, one has $\rho(E) = 1$ when $E \geq 0$, and $\rho(E) = 0$ when $E < 0$, which gives $\sigma_j \bar{\sigma}_k - \delta_j \bar{\delta}_k = 0$, and $\sigma_j \bar{\sigma}_k + \delta_j \bar{\delta}_k = 2e^{i(E_j-E_k)t/\hbar}$.

In general, the denominator is

$$\text{tr}[R P^\mu] = 2 \sum_{j=1}^N \gamma_j \bar{T}_{\mu k} T_{\mu j}.$$

Whenever $\rho(E) + \rho(-E) = 1$, as happens for Fock states and also the thermal states where $\rho(E) = (1 + e^{-\beta E})^{-1}$, we have $\gamma_j = 1$, and then the unitarity of T means that the denominator is 2, giving

$$\langle N_\lambda(t) \rangle_\mu = \frac{1}{2} \sum_{j,k=1}^N [(\sigma_j \bar{\sigma}_k + \delta_j \bar{\delta}_k)(1 + S_{jk}) + (\sigma_j \bar{\sigma}_k - \delta_j \bar{\delta}_k)(1 - S_{jk})] \bar{T}_{\lambda j} T_{\lambda k} \bar{T}_{\mu k} T_{\mu j}$$

and our earlier formulae for $\sigma_j \bar{\sigma}_k \pm \delta_j \bar{\delta}_k$ show that this contains both standard oscillations depending on the energy differences and others depending on $E_j + E_k$. In the Fock vacuum state the oscillation formula is consistent with the calculations performed in [29].

Taking $\lambda = \mu$, the oscillation formula takes on a slightly more compact form

$$\langle N_\mu(t) \rangle_\mu = \frac{1}{2} \sum_{j,k=1}^N [(\sigma_j \bar{\sigma}_k + \delta_j \bar{\delta}_k)(1 + S_{jk}) + (\sigma_j \bar{\sigma}_k - \delta_j \bar{\delta}_k)(1 - S_{jk})] |T_{\mu j}|^2 |T_{\mu k}|^2.$$

6. Non-unitary mixing in a Fock state

We could instead work with Ω the flavour vacuum. Then there is no need to inject W into $W \oplus W$, so that we have $S = 0$ and $R = 1$. This gives

$$\langle N_\lambda(t) \rangle_\mu = \frac{\text{tr}[F^\lambda P^\mu]}{\text{tr}[P^\mu]}$$

where

$$F^\lambda = u_f^* P^\lambda u_f - v_f^* P^\lambda v_f$$

with

$$u_f = a_{fm} V_t a_{mf} + b_{fm} V_t b_{mf} \quad v_f = a_{fm} V_t b_{mf} + b_{fm} V_t a_{mf}.$$

From this we may show that both sorts of oscillation terms occur in this case too. However, the total flavour number is given by replacing F^λ by $F = u_f^* u_f - v_f^* v_f$, and

$$\langle N_\lambda(t) \rangle_\mu = \frac{\text{tr}[F P^\mu]}{\text{tr}[P^\mu]}$$

and in general this depends on the time t . This is essentially a squeezing phenomenon. It provides a strong reason to be cautious about non-unitary mixing of this kind.

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Appendix

For a thermal state at temperature $(k\beta)^{-1}$ the KMS condition and anticommutation relations give formally

$$\begin{aligned} \omega_\beta[c(w)^* \tilde{D}c(z)] &= \omega_\beta[\tilde{D}c(z)c(e^{\beta H_D} w)^*] \\ &= \omega_\beta[\tilde{D}(\langle e^{\beta H_D} w, z \rangle - c(e^{\beta H_D} w)^* c(z))] \\ &= \langle e^{\beta H_D} w, z \rangle \omega_\beta[\tilde{D}] - \omega_\beta[c(e^{\beta H_D} w)^* \tilde{D}c(z)] + \omega_\beta[c(D e^{\beta H_D} w)^* c(z)]. \end{aligned}$$

This can be rearranged as

$$\omega_\beta[c((1 + e^{\beta H_D})w)^* \tilde{D}c(z)] = \langle e^{\beta H_D} w, z \rangle \omega_\beta[\tilde{D}] + \omega_\beta[c(D e^{\beta H_D} w)^* c(z)]$$

or, replacing w by $(1 + e^{\beta H_D})^{-1} w$,

$$\omega_\beta[c(w)^* \tilde{D}c(z)] = \langle (1 + e^{-\beta H_D})^{-1} w, z \rangle \omega_\beta[\tilde{D}] + \omega_\beta[c(D(1 + e^{-\beta H_D})^{-1} w)^* c(z)].$$

The case $\tilde{D} = 1$ (and $D = 0$) gives the usual two-point correlation function

$$\omega_\beta[c(w)^* c(z)] = \langle (1 + e^{-\beta H_D})^{-1} w, z \rangle$$

so that

$$\omega_\beta[c(w)^* \tilde{D}c(z)] = \langle (1 + e^{-\beta H_D})^{-1} w, z \rangle \omega_\beta[\tilde{D}] + \langle (1 + e^{-\beta H_D})^{-1} D(1 + e^{-\beta H_D})^{-1} w, z \rangle.$$

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